Exercise 28

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \frac{x^2 - 1}{2x - 3}$$

Solution

The domain of $f(x)$ is

$$2x - 3 \neq 0$$

$$2x \neq 3$$

$$x \neq \frac{3}{2}$$

$$\left\{ x \,\middle|\, x \neq \frac{3}{2} \right\}.$$

Calculate the derivative of $f(x)$ using the definition.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{(x+h)^2 - 1}{2(x+h) - 3} - \frac{x^2 - 1}{2x - 3}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x^2 + 2xh + h^2 - 1}{2x + 2h - 3} - \frac{x^2 - 1}{2x - 3}}{h} \\
&= \lim_{h \to 0} \frac{\frac{(2x-3)(x^2 + 2xh + h^2 - 1)}{(2x-3)(2x+2h-3)} - \frac{(x^2-1)(2x+2h-3)}{(2x-3)(2x+2h-3)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{(2x-3)(x^2 + 2xh + h^2 - 1) - (x^2-1)(2x+2h-3)}{(2x-3)(2x+2h-3)}}{h} \\
&= \lim_{h \to 0} \frac{(2x-3)(x^2 + 2xh + h^2 - 1) - (x^2-1)(2x+2h-3)}{h(2x-3)(2x+2h-3)} \\
&= \lim_{h \to 0} \frac{(\cancel{2x^3} + 4x^2h + 2xh^2 - 2x - \cancel{3x^2} - 6xh - 3h^2 + \cancel{3}) - (\cancel{2x^3} + 2x^2h - \cancel{3x^2} - 2x - 2h + \cancel{3})}{h(2x-3)(2x+2h-3)} \\
&= \lim_{h \to 0} \frac{(4x^2h + 2xh^2 - \cancel{2x} - 6xh - 3h^2) - (2x^2h - \cancel{2x} - 2h)}{h(2x-3)(2x+2h-3)} \\
&= \lim_{h \to 0} \frac{2x^2h + 2xh^2 - 6xh - 3h^2 + 2h}{h(2x-3)(2x+2h-3)}
\end{aligned}$$

Continue the simplification.

$$f'(x) = \lim_{h\to 0} \frac{h(2x^2 + 2xh - 6x - 3h + 2)}{h(2x-3)(2x+2h-3)}$$

$$= \lim_{h\to 0} \frac{2x^2 + 2xh - 6x - 3h + 2}{(2x-3)(2x+2h-3)}$$

$$= \frac{2x^2 - 6x + 2}{(2x-3)(2x-3)}$$

$$= \frac{2(x^2 - 3x + 1)}{(2x-3)^2}$$

The domain of $f'(x)$ is

$$(2x-3)^2 \neq 0$$

$$2x - 3 \neq 0$$

$$2x \neq 3$$

$$x \neq \frac{3}{2}$$

$$\left\{ x \;\middle|\; x \neq \frac{3}{2} \right\}.$$